Journal of
Mechanical
Science and
Technology

Journal of Mechanical Science and Technology 21 (2007) 821~828

Heuristic Study of Frictional-collisional Behavior for Granular Flow: A Continuum Approach

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(Manuscript Received August 28, 2006; Revised March 23, 2007; Accepted March 26, 2007)

Abstract

The frictional-collisional behavior of granular materials in motion under gravity is studied by using a continuum approach with a heuristic study on a collision mode. In a binary mode there are three basic mechanical motions, which are called kinetic, collisional, and frictional motion. They correspond to flying, contact, and friction time, respectively. Constitutive equations are then postulated with four constants, which are dispersive pressure, viscosity, thermal diffusivity, and collisional energy sink. The conservation laws for mass, momentum and energy are posed with the constitutive equations developed to study the granular flows on an inclined and in a vertical channel.

Keywords: Granular material; Continuum approach; Nanopowder; Constitutive equation

1. Introduction

Correct understanding of granular systems is in desperate demand not only in manufacturing processes but also in a wide spectrum of areas associated with small particles, such as grains and powders. State-of-the-art medical devices or medicine production, treatment of farming or dairy goods, snow avalanches or land slides, just to name a few, all will be greatly influenced and benefited by successful research on granular flows.

Many researchers have reported the development of explaining granular motions using continuum approach and produced models for granular behavior. Bagnold (1954) performed experiments on spherical particles suspended in Newtonian fluids, which has two limiting types of granular behavior. Goodman and Cowin (1971) devoted to a continuum theory under the steady-state, fully-developed and uniform depth for all channel inclinations greater than the angle of repose. Jenkins and Savage (1981) reported the granular fluctuation, called thermal velocity, analogous to fluid turbulence for rapid granular shear flow with a modified boundary-value problem.

Haff (1983) explored a general desire to improve the fundamental basis of the model which is an entirely heuristic application using microscopic approaches. Johnson and Jackson (1987) showed the constitutive relations combined with the contribution of the collisional and translational mechanism due to stress transmission with simplicity. Meanwhile, Hwang and Hutter (1995) reported the definition of two different kinds of times in a granular motion using a binary collision from Haff (1983) using the defined time of an encounter between two particles. In experiment, Azanza (1999) reported an experimental study about the flow of a slightly inelastic granular material on a rugged inclined plate and the prediction of a kinetic theory for dispersion and inelasticity.

In this study we introduce a new model developed from that of Haff (1983) and of Hwang and Hutter

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(1995). The effects of all three possible states of particles are included, which are kinetic, colli-sional, and frictional motion. These three processes induce mutually distinctive behaviors, with three different time scales. We show three motions of binary particles, and provide the parameterization of these motions. Using the parameterization of the three motions, dispersive pressure, viscosity, thermal diffusivity and collisional energy sink in microscopic model are derived. Some applications to granular flows under gravity are examined.

2. Three phenomena of a binary particle movement

The movements of granular flow are associated with interactions among adjacent particles. Among the particles in the flow system, we will look into the movement of neighboring particles by picking two neighboring particles in Fig. 1. The movement consists of three different motions; kinetic, collisional, and frictional motion, while particles are perfectly spherical and their surface is rugged. To understand the binary collision, we should know the possible interactions between two particles. The particles move forward and backward within the distance of neighboring particles. This motion can be called 'approaching,' where the bulk velocity of granular media is provided. After approaching, collision of two particles occurs. The two particles make contact with each other, and then the motion is called 'contact'. This motion can be described as the energy transportation between the two particles. The elasticity of a particle is involved in the motion. During the colliding process they completely give away their kinetic energy which is transferred within the particle into vibrating wave energy. Once the elastic wave within a particle is reflected and refracted at the boundary and returned to the contact point of the other particle, kinetic energy is created again. Velocity fluctuations in the motion cause the exchange of momentum between adjacent layers as fluid-like flow. The last motion appears in case of eccentric interaction between the two particles. It produces time delay due to the spinned or rubbed particles, giving away their fictional energy which is transferred within the particle into shear stress energy. Once the shear stress within the two particles returns to the contact point of two particles, frictional energy will be created again. This motion can be regarded as a factor of shear stress between an upper particle and a lower particle. It thus appears that there are three independent motions in a granular flow, as stated earlier.

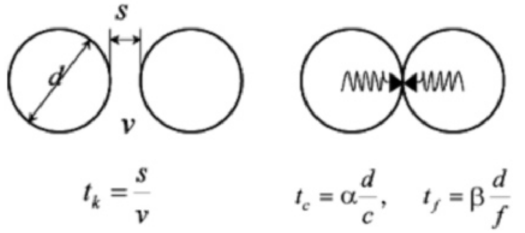

Fig. 1. Two identical spheres in microscopic model.

3. Parameterization of three phenomena

More explicit descriptions can be given to the physical effect corresponding to two particles sketched in Fig. 1. Our goal is to extract reasonable parameters representing three states of granular behavior with their characteristics. Three different times are used at a particular configuration. Let t_k, t_c and t_f be kinetic time (approaching motion to neighbors), collision time (contact motion) and frictional time (friction motion), respectively, which are defined as

$$t_k = \frac{s}{v}, \ t_c = \alpha\frac{d}{c}, \ t_f = \beta\frac{d}{f}. \tag{1}$$

Here t_k and t_c are as introduced also by Haff (1983) and Hwang and Hutter (1995). Kinetic time t_k is associated with separation distance and fluctuation velocity. Large t_k caused by s large separation distance or small v implies non-stationary motion of granular flow, which corresponds to the classical fluid mechanics. In the kinetic theory of gases, thermal speed of transfer is represented by c which is a statistical quantity in this model. If $t_c = 0$ is satisfied with $c = 0$ which results in large elasticity of the particle and corresponds to rigid particles of motion. In the model, t_f is a new scale used in this study. t_f shows the terms of the surface condition of a particle and the strength of shear stress from friction in turbulence. The total shear stress is given by $\tau/\rho = \nu(du/dy) - u'v'$. The term τ/ρ frequently arises in the consideration of turbulent flows for due to the friction of the fluid, and has a dimension of velocity squared. The quantity $\sqrt{\tau/\rho}$ is called friction velocity. If $t_f = 0$ is satisfied with $f = 0$ which corresponds large shear stress inbetween the particles, and result in the limit of rapid granular flow. Based on Reynolds stress, $\sqrt{\tau/\rho}$ is applicable for the friction velocity to the behavior of granular flows to obtain frictional time. Each time equations work independently, which result that the previous theoretical cases are automatically satisfied with when $\beta = 0$ or $\alpha = \beta = 0$ is applied.

4. The equation of motion

We have a granular system described by the three conservation equations of classical continuum mechanics (Haff, 1983):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0, \tag{2}$$

$$\frac{\partial}{\partial t}(\rho u_i) = -\frac{\partial}{\partial x_k}\left[p\delta_{ik} + \rho u_i u_k - \eta\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\right] + \rho g_i, \tag{3}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x_k}\left[\rho u_k\left(\frac{p}{\rho} + \frac{1}{2}u^2 + \frac{1}{2}v^2\right)\right] = \frac{\partial}{\partial x_k}\left[u_i\eta\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) + k\frac{\partial}{\partial x_k}\left(\frac{1}{2}\rho v^2\right)\right] + \rho u_i g_i - I \tag{4}$$

The pressure p and the viscosity η are as widely used in a classical fluid mechanics. The fluctuation velocity (v) is as used by Ogawa (1978) and Cowin (1979). Since we have partitioned the energy into two parts, overall flow kinetic energy and internal energy, it is seen that internal energy tends to behave like temperature of the system, where internal energy per particle is characterized by a Maxwell-Bolzmann form with kT. The quantity of I gives the rate at which energy is lost from the system and transformed into heat due to the grain-grain collision. As described, p, η, k, and I should be established depending on the collision rate when granular materials flow.

5. The microscopic model

The dispersive pressure (p) can be established dimensionally. It is defined as "force per area" or "mass times acceleration per area." The state in kinetic time and the state in collision time and frictional time have different mass quantities because no separation distance exists during collision time and friction time comparing with kinetic time, so that the mass corresponding to kinetic time is $m = \rho(d+s)^3$, and the mass corresponding to the collisional time and frictional time is $m = \rho(d)^3$. The area of a cell is $(d+s)^2$ for the motion of kinetic time and d^2 for the motion of collision and friction time. Since the states of each times are independent, the expression of dispersive pressure can be obtained as

$$\begin{aligned} p &= t\left(\rho(d+s)\frac{v}{t_k} + \rho d\frac{v}{t_c} + \rho d\frac{v}{t_f}\right) \\ &= t\rho\left((d+s)\frac{v^2}{s} + \frac{vc}{\alpha} + \frac{vf}{\beta}\right) \end{aligned} \tag{5}$$

where t is a dimensionless constant.

It is the simplest way of considering two adjacent layers of particles. When an independent motion occurs between two layers due to the difference of the mean velocity among neighboring particles, the dynamics viscosity can be dimensionally given by "density times area divided by time." If ρ is the density, the state of viscosity can be obtained as

$$\sigma = q\rho\left(\frac{(d+s)^2}{t_k} + \frac{d(d+s)}{t_c} + \frac{d(d+s)}{t_f}\right)\frac{du}{dy} \tag{6}$$

where $\delta u/(d+s) \sim du/dy$ in kinetic time and $\delta u/d \sim du/dy$ in the collision and friction time. q is a dimensionless constant. The viscosity coefficient (η) is then expressed as

$$\eta = q\rho\left(\frac{(d+s)^2 v}{s} + \frac{(d+s)c}{\alpha} + \frac{(d+s)f}{\beta}\right) \tag{7}$$

It makes sense that the effect of the dispersive pressure effect appears more significant than the effect of viscosity when pressure depends on bulk viscosity in the microscopic model.

Thermal diffusivity coefficient (k) in energy equation can be found in a similar dimension as energy flux due to grain-grain collision is averagely net energy transfer with respect to collision times collision rate divided by area. Thermal diffusivity appears in the dimensional similarity with the kinetic viscosity, so that the simple postulation of thermal diffusivity is shown as $k \sim \eta / \rho$, where r is a constant:

$$k = r\left(\frac{(d+s)^2}{t_k} + \frac{d(d+s)}{t_c} + \frac{d(d+s)}{t_f}\right) \tag{8}$$

Since the grain-grain collision is inelastic, the grain system loses its own energy, where energy is lost through collisions per unit volume per time. It thus can be expressed as $\Delta E \sim (1-e^2)mv^2/2$, where e is the coefficient of restitution and m is the mass of the colliding particles. The multiplication of the collision rate (v/s) and the density of the particles yields

$$I = \gamma\rho\left(\frac{v^2}{t_k} + \frac{v^2}{t_c} + \frac{v^2}{t_f}\right), \tag{9}$$

where γ is constant proportional to $1-e^2$ (Ogawa, 1978).

6. Some applications of the model developed

6.1 Steady-state granular flow on an inclined plate

Consider a granular flow on an inclined plate with a certain angle against the horizontal, as shown in Fig. 2. It is a fully-developed steady flow with a uniform depth, and variables depend only on the y coordinate. Boundary conditions are adopted from previous works for faithful comparisons. No slip condition is applied at the bottom of the plate, as in Anderson (1992), Savage (1979) and Azanza (1999). Zero thermal velocity at the bottom and zero quantity of derivative of thermal velocity at free surface are applied as in Anderson (1992) and Azanza (1999). The derivative of thermal velocity is forced to vanish at infinity, which ensures the nullity of stress and the flux of fluctuation kinetic energy at the free surface. The boundary conditions then are expressed as

$$u_{bottom} = 0\,,\ \ v_{bottom} = 0\,,\ \ \left.\frac{dv}{dy}\right|_{top} = 0 \tag{10}$$

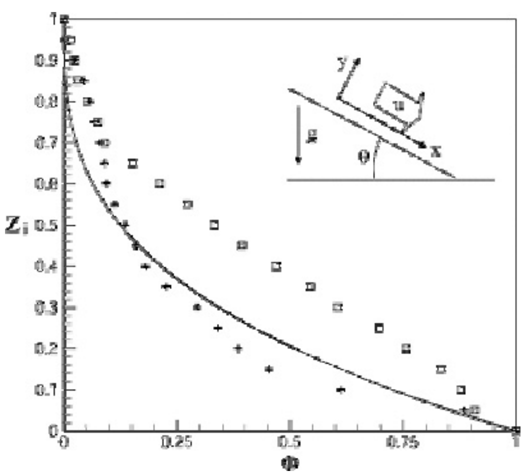

Fig. 2. Dimensionless volume fraction of granular flow on an inclined plate, where $\theta = 22$ which is the same with Anderson (1992), Savage (1979) and Azanza (1999). □:Theoretical result (Anderson,1992), and +:Experimental result (Azanza,1999). All constants are set 1.

Before solving the problem of the granular flow on an inclined plate, we should choose a constitutive equation for the volume fraction. Savage (1979) introduced the volume fraction depending on the interaction between shear stress and normal stress when particles flow, and used a constitutive equation for the volume fraction as described by Bagnold (1954) as

$$\frac{h-y}{h} \approx \left[\frac{\phi}{\phi^*}\right]^{1/3}, \tag{11}$$

where ϕ is volume fraction of granular flow and ϕ^* is volume fraction of the granular flow at a position. We apply this equation to the present model.

The continuity equation is satisfied automatically. The conservation of momentum for y coordinate and x for coordinate can be obtained, respectively, as

$$p(y) = \rho g \cos\theta (h-y) + p_0\,, \tag{12}$$

$$\sigma(y) = \eta\frac{du}{dy} = \rho g \sin\theta (h-y) + \sigma_0 \tag{13}$$

where p_0 is dispersive pressure at free surface and σ_0 is shear stress at surface. The conservation of energy equation with Eq. (11) can be obtained as

$$\frac{\sigma^2(y)}{\eta} + \frac{\gamma}{t}\left[\frac{ds}{dy}p\frac{dv}{dy} - (d+s)\rho g\cos\theta\frac{dv}{dy} + (d+s)p\frac{d^2v}{dy^2}\right] = 0 \tag{14}$$

The dimensionless parameters in the system are taken as $z_i = y/h$, $V = v/\bar{v}$, and $U = u/\bar{u}$, where $\bar{v}$ and $\bar{u}$ are maximum value.

Applying Eq. (12) and Eq. (13) into Eq. (14) with the dimensionless parameters, we can obtain an equation for thermal velocity as

$$\frac{d^2V}{dz_i^{\,2}} + T^2 z_i^{\,2} V = 0 \quad \left(T^2 = \frac{t}{qe^2\gamma}\phi^{*2/3}\tan^2\theta \right) \qquad (15)$$

Equation (15) can be solved by a fourth-order Runge-Kutta method and a shooting method on the derivative of the thermal velocity at the bed. The flow velocity can be obtained from Eq. (13) after solving the dimensionless thermal velocity. All constants are set to unity for convenience.

Figure 2 shows the dimensionless volume fraction comparing with theoretical result from Anderson (1992) and experimental result from Azanza (1999) on an inclined plate, where $\theta = 22$, the same value with Anderson (1992) and Azanza (1999). The maximum value of the volume fraction appears at bottom and the minimum value of the volume fraction appears at free surface. The theoretical result linearly decreases from the bottom while the experimental result shows quadratic decrease from the bottom. Present result agrees with the experimental result better than the theoretical result.

Figure 3 shows dimensionless flow velocity on an inclined plate in comparison with the theoretical results of Anderson (1992) and Savage (1979) and the experimental result by Azanza (1999). It is interesting to note that the flow velocity is different from a classical fluid flow. The gradient of the velocity decreases from the bottom to the mid point of the depth, while the gradient of the velocity increases beyond the mid point. It can be expected that normal stress effect from the mid-point depth to the bottom becomes pronounced, which causes the flow velocity to decrease gradually with respect to lower flow, while shear stress effect from the mid-point depth to the free surface becomes pronounced, which causes the flow velocity to increase gradually with respect to upper flow. All results in Fig. 3 are qualitatively similar, but the present result agrees better with the experimental result.

Figure 4 shows the dimensionless thermal velocity on an inclined plate in comparison with the theoretical results of Anderson (1992) and experimental result of Azanza (1999). While thermal velocity strongly depends on the volume fraction, the dimen-

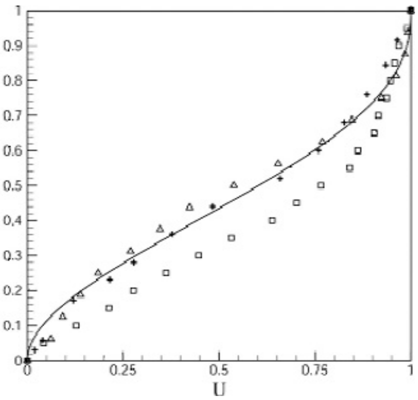

Fig. 3. Dimensionless granular flow velocity on an inclined plate comparing with experimental and theoretical results, where $\theta = 22$. □:Theoretical result (Anderson,1992), Δ : Theoretical result (Savage,1979), and +: Experimental result (Azanza,1999).

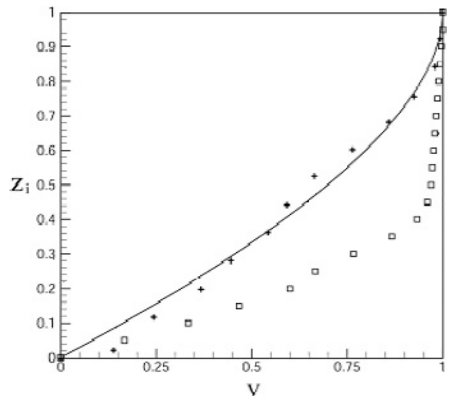

Fig. 4. Thermal velocity of granular flow on an inclined plate comparing with experimental and theoretical results, where $\theta = 22$. □: Theoretical result (Anderson,1992), and +:

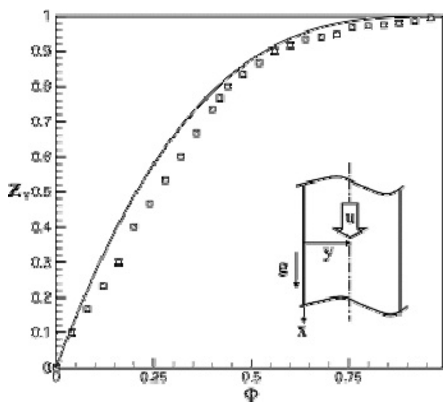

Fig. 5. Dimensionless volume fraction of granular flow in a vertical channel comparing with theoretical results. □: Theoretical results (Savage,1998). All constants are set 1.

sionless thermal velocity shows quadratic dependence. All results appear in qualitative agreement, while the present result again shows better simulation of the experimental result.

6.2 Steady-state granular flow in a vertical channel

Consider the granular flow in a vertical channel as shown in Fig. 6. It is fully developed and steady under gravity, and variables depend only on the y coordinate. The boundary conditions are taken from the previous work; zero flow velocity at the wall referred from Savage (1998), zero derivative of thermal velocity at the center of the channel referred from Savage (1998, 1979), and the thermal velocity at the wall is zero as referred to Savage (1998). They are

$$u_{wall} = 0\ ,\ \ v_{wall} = 0\ ,\ \ \left.\frac{dv}{dy}\right|_{center} = 0 \tag{16}$$

The continuity equation is satisfied automatically. The conservation of momentum equation for y coordinate is $p = p_0$ and for x coordinate is

$$\sigma(y) = \eta\frac{du}{dy} = \rho g \sin\theta(h-y) + \sigma_0 \tag{17}$$

The conservation of energy equation is obtained as

$$\frac{\sigma^2(y)}{\eta} + \frac{\gamma}{t}\left[\frac{ds}{dy}p_0\frac{dv}{dy} + (d+s)p_0\frac{d^2v}{dy^2}\right] = 0 \tag{18}$$

With Eqs. 17~18, the thermal velocity can be obtained as

$$\frac{d^2V}{dz_v^{\ 2}} - \frac{1}{z}\frac{dV}{dz_v} + Ww^2z_v^{\ 4}V = 0\ ,$$
$$\left(W = \frac{\phi^{*2/3}}{4e^2}\frac{t^2}{q\gamma}, w = \frac{2\delta}{d}\right) \tag{19}$$

where $z_v = y/\delta$.

Figure 5 shows dimensionless volume fraction in a vertical channel and comparison with theoretical result from Savage (1998). The right margin of the figure coincides with the center of the vertical channel, while the left margin is the wall of the channel. It is interesting to note that the behavior of granular flow in a vertical channel is considerably different from the behavior of granular flow on an inclined plate.

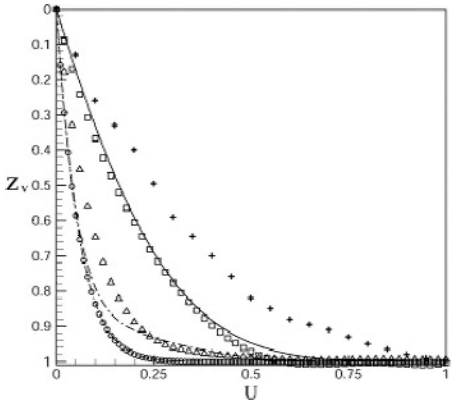

Fig. 6. Dimensionless granular flow velocity in a vertical channel comparing with theoretical and experimental results for $w = 20(-)$, $w = 80(-\cdot-)$ and $w = 1000(-\cdot\cdot-)$. Theoretical results for $w = 20$ (□), 80(Δ), and 1000(○) (Savage, 1998). Experimental result for $w = 80$ (+) (Savage, 1979).

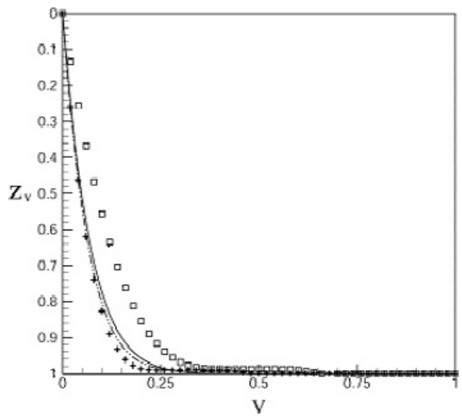

Fig. 7. Dimensionless thermal velocity of granular flow in a vertical channel comparing with theoretical and experimental results for $w = 20(-)$ and $w = 80(-\cdot\cdot-)$.:Theoretical result with $w = 20$ (□) and $w = 80$ (+) (Savage, 1998).

Minimum volume fraction appears at the wall, while maximum volume fraction in granular flow on an inclined plate appears at the bottom in Fig. 2. It means that the wall effect for volume fraction becomes weak, so that the normal stress becomes gradually dominated far from the wall, and the maximum volume fraction appears at the center, and the flow behaves as bulk-like flow because there is less interaction between neighboring particles.

Figure 6 shows dimensionless flow velocity in vertical flow and comparisons with the theoretical result of Savage (1998) with $w = 20$, 80 and 1000 and the experimental result of Savage (1979) with $w = 80$. Since the normal stress becomes gradually dominated, the pattern of flow velocity is similar to the fluid flow velocity. The flow behaves like a bulk-like flow in the center of the vertical channel. As w increases, the plug-flow region spreads.

Figure 7 shows the dimensionless thermal velocity in vertical channel in comparison with theoretical result from Savage (1998) with $w = 20$ and 80. As thermal velocity reversely depends on the volume fraction, small thermal velocity appears in large volume fraction, while large thermal velocity appears in small volume fraction.

7. Concluding remarks

We studied the frictional-collisional behavior of granular materials in motion under gravity by using a continuum approach. Three processes, kinetic, collision, and friction motion, are modeled, which are compared favorably against previous results in terms of the volume fraction, the thermal velocity, and the flow velocity. The equations developed contain the effects of stress, viscosity, thermal diffusivity, and energy sink, and we discussed the variation of parameters: p , η , k and I . We defined the four factors with physical definition, formulated definition, and then obtained the volume fraction, thermal velocity, and flow velocity.

As the separation distance increases, the effects of dispersive pressure, viscosity, thermal diffusivity, and energy sink seem less pronounced, while those of the granular elasticity and granular surface properties stand out. The effect of the plate in the inclined plate flow and the effect of the wall in vertical channel flow produced different results for the thermal velocity problem. The large thermal velocity induced positive effect in the inclined plate flow, while the small thermal velocity induced the positive effect in vertical channel flow, which explains why the plug-like flow occurs at the center of the channel.

Nomenclature

c	:	The elastic speed of granular media
d	:	Particle diameter
f	:	The frictional speed of granular media
g	:	Gravitational acceleration
h	:	Depth of granular flow on an inclined flow
i	:	The collisional energy sink
k	:	Thermal diffusivity
p	:	Dispersive pressure
s	:	The mean separation distance pf particles
t_k	:	Kinetic time
t_c	:	Collisional time
t_f	:	Frictional time
u	:	Flow velocity
U	:	Dimensionless particle flow velocity
v	:	Approaching speed of granular media
V	:	Dimensionless approaching speed of granular media
α	:	Dimensionless numbers of order unity
β	:	Dimensionless number of order unity
δ	:	Half-width of vertical channel
ρ	:	Density of granular media
η	:	Viscosity
ϕ	:	Volume fraction
ϕ^*	:	Maximum volume fraction
ϕ_b	:	Volume fraction at a solid plate
Φ	:	Dimensionless volume fraction ($= (\phi - \phi_b)/(\phi^* - \phi_b)$)
θ	:	Angle of an inclined plate
w	:	Dimensionless width of a vertical channel

Acnowlegement

This work is supported by the National Center for Nanomaterials Technology of Pohang through Yeungnam University.

8. Reference

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